15.0 - Introduction

This chapter will give you a new take on the saying, “What goes around comes around.”

An oscillation is a motion that repeats itself. There are a myriad of examples of oscillations: a child playing on a swing, the motion of the Earth in an earthquake, a car bouncing up and down on its shock absorber, the rapid vibration of a tuning fork, the diaphragm of a loudspeaker, a quartz in a digital watch, the amount of electric current flowing in certain electric circuits, etc.!

Motion that repeats itself at regular intervals is called periodic motion. A traditional metronome provides an excellent example of periodic motion: Its periodic nature is used by musicians for timing purposes. Simple harmonic motion (SHM) describes a specific type of periodic motion. SHM provides an essential starting point for analyzing many types of motion you often see, such as the ones mentioned above.

SHM has several interesting properties. For instance, the time it takes for an object to return to an endpoint in its motion is independent of how far the object moves. Galileo Galilei is said to have noted this phenomenon during an apparently less-than-engrossing church service. He sat in the church, watching a chandelier swing back and forth during the service, and noticed that the distance the chandelier moved in its oscillations decreased over time as friction and air resistance took their toll. According to the story, he timed its period — how long it took to complete a cycle of motion — using his pulse. To his surprise, the period remained constant even as the chandelier moved less and less. (Although this is a well-known anecdote, apparently the chandelier was actually installed too late for the story to be true.)

To begin your study of simple harmonic motion, you can try the simulation to the right. A mass (an air hockey puck) is attached to a spring, and glides without friction or air resistance over an air hockey table, which you are viewing from overhead. When the puck is pushed or pulled from its rest position and released, it will oscillate in simple harmonic motion.

A pen is attached to the puck, and paper underneath it scrolls to the left over time. This enables the system to produce a graph of displacement versus time. A sample graph is shown in the illustration to the right. A mass attached to a spring is a classic configuration used to explain SHM, and the graph of the mass’s displacement over time is an important element in analyzing this form of motion.

Using the controls, you can change the amplitude and period of the puck’s motion. The amplitude is the maximum displacement of the puck from its rest position. The period is the time it takes the puck to complete one full cycle of motion.

As you play with the controls, make a few observations. First, does changing the amplitude change the period, or are these quantities independent? Second, does the shape of the curve look familiar to you? To answer this question, think back to the graphs of some of the functions you studied in mathematics courses.

15.1 - Simple harmonic motion

Simple harmonic motion: Motion that follows a repetitive pattern, caused by a restoring force that is proportional to displacement from the equilibrium position.

At the right, you see an overhead view of an air hockey table with a puck attached to a spring. Friction is minimal and we ignore it. The only force we concern ourselves with is the force of the spring on the puck.

Initially, the puck is stationary and the spring is relaxed, neither stretched nor compressed. This means the puck is at its equilibrium (rest) position. Imagine that you reach out and pull the puck toward you. You see this situation in Concept 1 to the right.

Now, you release the puck. The spring pulls on the puck until it reaches the equilibrium point. At this point, the spring exerts no force on the puck, since the spring is neither stretched nor compressed. As it reaches the equilibrium point, the puck’s speed will be at its maximum. You see this in Concept 2.

The puck’s momentum means it will continue to move to the left beyond the equilibrium point. This compresses the spring, and the force of the spring now slows the puck until it stops moving. You see this in Concept 3. At this point, the puck’s velocity is zero.

Both the displacement of the puck from the equilibrium position and the force on it are now the opposites of their starting values. The puck is as far from the equilibrium point as it was when you released it, but on the opposite side. The spring exerts an equal amount of force on the puck as it initially did, but in the opposite direction. The force will start to accelerate the puck back to the right.

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The motion continues. The spring expands, pushing the puck to the equilibrium point. The puck passes this point and continues on, stretching the spring. It will return to the position from which you released it. There, the force of the stretched spring causes the puck to accelerate to the left again. Without any friction or air resistance, the puck would oscillate back and forth forever.

As you may have noted, “equilibrium” means there is no net force present. It does not mean “at rest” since the puck is moving as it passes through the equilibrium position. It is where the spring is neither stretched nor compressed.

The motion of the puck is called simple harmonic motion (SHM). The force of the spring plays an essential role in this motion. Two aspects of this force are required for SHM to occur. First, the spring exerts a restoring force. This force always points toward the equilibrium point, opposing any displacement of the puck. This is shown in the diagrams to the right: The force vectors point toward the equilibrium position.

Second, for SHM to occur, the amount of the restoring force must increase linearly with the puck’s displacement from the equilibrium point. Why can a spring cause SHM? Springs obey Hooke’s law, which states that \( F = -kx \). The factor \( k \) is the spring constant and it does not vary for a given spring. As \( x \) (the displacement from equilibrium) increases in absolute value, so does the force. For instance, as the puck moves from \( x = 0.25 \) m to \( x = 0.50 \) m, the amount of force doubles. In sum, since a spring causes a restoring force that increases linearly with displacement, it can cause SHM.

We have extensively used the example of a puck on an air hockey table here, but this is just one configuration that generates SHM. For example, we could also hang the puck from a vertical spring and allow the puck’s weight to stretch the spring until an equilibrium position was reached. If the puck were then pulled down from this position, it would oscillate in SHM, since the net force on the puck would be proportional to its displacement from equilibrium but opposite in sign.

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15.2 - Simple harmonic motion: graph and equation

At the right, the puck is again moving in SHM, and a graph of its motion is shown. In this case, we have changed our view of the air hockey table so the puck moves vertically instead of horizontally. This puts the graph in the usual orientation. We continue to measure the displacement of the puck with the variable \( x \), which is plotted on the vertical axis. The horizontal axis is the time \( t \).

Unrolling the graph paper underneath the puck as it moves up and down would create the graph you see, the blue line on the white paper. The graph traces out the displacement from equilibrium of the puck over time as it moves from “peaks” where its displacement is most positive, to “troughs” where it is most negative. It starts at a peak, passes through equilibrium, moves to a trough, and so on. After four seconds, it has returned to its initial position for the second time.

The graph might look familiar to you. If you have correctly recognized the graph of a cosine function, congratulations! A cosine function describes the displacement of the puck over time. You see this function in Equation 1.

This graph represents the puck starting at its maximum displacement. When \( t = 0 \) seconds, the argument of the cosine function is zero radians and the cosine is one, its maximum value. (In describing SHM, the units of the argument of the cosine must be
Because the function used for this graph multiplies the cosine function by an amplitude of three meters, the maximum displacement of the puck (this is always measured from equilibrium) is also three meters.

Equation 2 shows the general form of the equation for SHM. The parameters \( A \), \( \omega \), and \( \phi \) are called the amplitude, angular frequency, and phase constant, respectively. The argument of the cosine function, \( \omega t + \phi \), is called the phase. In sections that follow we will explain how these parameters are used to describe SHM.

**15.3 - Period and frequency**

*Period:* Time to complete one full cycle of motion.

*Frequency:* Number of cycles of motion per second.

The period specifies how long it takes an object to complete a full cycle of motion. The letter \( T \) represents period, which is measured in seconds. A convenient way to calculate the period is to measure the time interval between two adjacent peaks, as we do in Equation 1. In the example shown there, it takes two seconds for the puck to complete one full cycle of motion.

The frequency, represented by \( f \), specifies how many cycles are completed each second. It is the reciprocal of the period. The graph in Equation 2 is the same as in Equation 1. Its frequency is 0.5 cycles per second.

Frequency has its own units. One cycle per second equals one hertz (Hz). This unit is named after the German physicist Heinrich Hertz (1857–1894). You may be familiar with the hertz units from computer terminology: The speed of computer microprocessors used to be specified in megahertz (one million internal clock cycles per second) but microprocessors now operate at over one gigahertz (one billion cycles per second). Radio stations are also known by their frequencies. If you tune into an AM station shown on the dial at 950, the frequency of the radio waves transmitted by the station is 950 kHz.
Angular frequency: Frequency measured in radians/second.

In the equation for SHM shown in Equation 1, the parameter \( \omega \) is the angular frequency and it is the coefficient of time in the equation for SHM. Its units are radians per second.

The angular frequency equals \( 2\pi \) times the frequency. The relationship between frequency and period can be used to restate this equation in terms of the period. Both these equations are shown in Equation 2.

You may have noticed that \( \omega \) also stands for the angular speed of an object moving in a circle, which is measured in radians per second, as well. If an object makes a complete loop around a circle in one second, its angular speed will be \( 2\pi \) radians per second. Similarly, an object in SHM that completes a cycle of motion in one second has an angular frequency of \( 2\pi \) radians per second. This is indicative of a relationship between circular motion and SHM that can be productively explored elsewhere.
Amplitude: Maximum displacement from equilibrium.

The amplitude describes the greatest displacement of an object in simple harmonic motion from its equilibrium position.

In Concept 1, you see the now familiar air hockey puck and spring, as well as a graph of its motion. The amplitude is indicated. It is the farthest distance of the puck from the equilibrium point.

The equation for SHM is shown again in Equation 1, with the amplitude term highlighted. The amplitude is the absolute value of the coefficient of the cosine function. The letter $A$ stands for amplitude. Since the amplitude represents a displacement, it is measured in meters.

Why does the amplitude equal the factor outside the cosine function? The values of the cosine range from $+1$ to $-1$. Multiplying the maximum value of the cosine by the amplitude (for example, four meters for the function shown in Example 1) yields the maximum displacement.

$T =$ period

$f =$ frequency

The function $x(t) = A \cos(\omega t)$ describes this graph. What is the angular frequency, $\omega$?

$\omega = \frac{2\pi}{T}$

$T = 3.5$ seconds

$\omega = \frac{2\pi}{3.5}$

$\omega = 1.8$ rad/s

Amplitude

Maximum displacement from equilibrium

Amplitude

$|A|$ units: meters (m)
What is the amplitude?
Amplitude = $|A| = 4 \text{ m}$

15.6 - Interactive problem: match the curve

In the simulation on the right, you control the amplitude and period for a puck on a spring moving in simple harmonic motion. With the right settings, the motion of the puck will create a graph that matches the one shown on the paper.

Determine what the values for the amplitude and period should be by examining the graph. Assume you can read the graph to the nearest 0.1 m of displacement and the nearest 0.1 s of time, and set the values accordingly. Press GO to start the action and see if your motion matches the target graph. If it does not, press RESET to try again.

Review the sections on amplitude and period if you have difficulty solving this problem.

15.7 - Phase and phase constant

**Phase:** Argument of the trigonometric function.

**Phase constant:** The term of the phase that determines the value of the function at the initial time.

In the general equation for SHM shown in Equation 1, the argument of the cosine function, $\omega t + \varphi$, is called the phase of the function.

The constant term $\varphi$ is called the phase constant. The phase constant can be determined by noting the displacement when $t = 0$ s and setting the phase constant appropriately given the function’s amplitude. For instance, if the amplitude is 2 meters and the displacement at $t = 0$ s is 1 meter, then the phase constant must be $\pi/3$ radians ($60^\circ$), since the cosine of this angle is 1/2. To check this, note that $(2 \text{ m}) \cos(0 + \pi/3) = 1 \text{ m}$.

Often, it is convenient to say the object starts at its maximum positive displacement $A$ at $t = 0$ s so that $\varphi = 0$ rad. In this chapter, we often choose examples where the phase constant is 0, to simplify the mathematical expressions.

If $\varphi$ is not zero, the graph is shifted to the left or right without changing its shape. For example, if we wanted the function $x(t)$ to start at zero displacement (that is, at equilibrium), we could use a phase constant of $\pi/2$ radians ($90^\circ$). You see this illustrated in Equation 1. Because of the periodic nature of the cosine, you can select a phase constant in the range 0 to $2\pi$ rad to achieve any desired shift of the function. For instance, there is no difference between a phase constant of 0.5 radians and one of $2.5\pi$ radians.

**Phase and phase constant**

$x(t) = A \cos (\omega t + \varphi)$

- **Phase** is $\omega t + \varphi$
- **Phase constant** is $\varphi$

**Units for $\varphi$:** radians

Given a graph of a simple harmonic motion function, you can calculate the phase constant. You see an example of this in Example 1. There will be two phase constants between 0 and $2\pi$ radians that result in the same displacement at $t = 0$ s, because of the identity $\cos (a) = \cos (2\pi - a)$. How can you choose between them? In the example, whether the phase constant equaled 1.1 or 5.2 radians, the graph would start at the correct initial value. To select which one is correct, you have to note whether the graph rises or falls after $t = 0$ s.

The cosine function decreases between 0 and $\pi$ radians, and increases between $\pi$ and $2\pi$ radians. Since the function shown in Example 1 is increasing at time $t = 0$ s, the phase constant is between $\pi$ and $2\pi$, so 5.2 is the correct choice.
An object moving in SHM has amplitude 6.3 m. At time \( t = 0 \) s, its displacement is 2.9 m. What is the phase constant, \( \phi \)?

\[
x(t) = A \cos (\omega t + \phi)
\]

\[
x(0 \text{ s}) = 2.9 \text{ m}
\]

\[
2.9 \text{ m} = (6.3 \text{ m}) \cos (\phi(0 \text{ s}) + \phi)
\]

\[
\cos \phi = (2.9 \text{ m})/(6.3 \text{ m})
\]

\[
\cos \phi = 0.46
\]

\[
\phi = 1.1 \text{ rad or } 5.2 \text{ rad}
\]

Because the graph increases at \( t = 0 \), \( \phi = 5.2 \text{ rad} \)

### 15.8 - Sample problem: graph equation

You see a graph of the displacement of an object moving in SHM. Determine the amplitude, period and phase constant, and use that information to write an equation for the displacement of the object as a function of time.

### Variables

- amplitude: \( A \)
- period: \( T \)
- angular frequency: \( \omega \)
- displacement: \( x \)
- time: \( t \)
- phase constant: \( \phi \)

### What is the strategy?

1. Determine the amplitude \( A \) and period \( T \) by examining the graph. From the period, calculate the angular frequency \( \omega \).
2. Examine the graph to determine the displacement at time \( t = 0 \), and use this and the graph to calculate the phase constant.
3. Use these values to write the SHM equation for the displacement of the object as a function of time.

### Physics principles and equations

The period and angular frequency are related by

\[
\omega = 2\pi/T
\]

The equation for the displacement of an object in SHM is

\[
x(t) = A \cos (\omega t + \phi)
\]
Step-by-step solution

The amplitude can be read directly off the graph. To determine the period, we examine the two peaks shown.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$A = 4.0 \text{ m}$ from graph</td>
</tr>
<tr>
<td>2.</td>
<td>$T = 2.0 \text{ s}$ from graph</td>
</tr>
<tr>
<td>3.</td>
<td>$\omega = 2\pi/T$ equation for angular frequency</td>
</tr>
<tr>
<td>4.</td>
<td>$\omega = 2\pi/2.0 = \pi$ enter value from step 2</td>
</tr>
</tbody>
</table>

Now we examine the graph to determine the displacement at time $t = 0 \text{ s}$, and use that to calculate the phase constant. We will need to select a phase constant so that the graph decreases after $t = 0 \text{ s}$.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.</td>
<td>$x(0 \text{ s}) = 0.0 \text{ m}$ from graph</td>
</tr>
<tr>
<td>6.</td>
<td>$x(t) = A \cos (\omega t + \phi)$ SHM displacement equation</td>
</tr>
<tr>
<td>7.</td>
<td>$0.0 \text{ m} = (4.0 \text{ m}) \cos \phi$ from steps 1, 5, and 6</td>
</tr>
<tr>
<td>8.</td>
<td>$\cos \phi = 0$ solve equation 7</td>
</tr>
<tr>
<td>9.</td>
<td>$\phi = \pi/2$ or $3\pi/2 \text{ rad}$ two possible solutions</td>
</tr>
<tr>
<td>10.</td>
<td>$\phi = \pi/2 \text{ rad}$ graph decreasing at $t = 0$</td>
</tr>
</tbody>
</table>

Finally, we put the pieces together to write the equation.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>11.</td>
<td>$x(t) = A \cos (\omega t + \phi)$ SHM displacement equation</td>
</tr>
<tr>
<td>12.</td>
<td>$x(t) = 4.0 \cos (\pi t + \pi/2)$ enter values</td>
</tr>
</tbody>
</table>

15.9 - Velocity

In Concept 1, you see a graph of an object in simple harmonic motion. The graph shows the displacement of the object versus time. At any point the slope of the graph is the object’s instantaneous velocity. The slope equals $\Delta y/\Delta t$. In this graph, this is the change in displacement per unit time, which is velocity.

You can consider the relationship of velocity and displacement by reviewing the role of force in SHM. Consider an object attached to a spring, where the spring is stretched and then the object is released. The spring force pulls the object until it reaches the equilibrium point, increasing the object’s speed.

Once the object passes through the equilibrium point, the spring is compressed and its force resists the object’s motion, slowing it down. Because the object speeds up as it approaches the equilibrium point and slows down as it moves away from equilibrium, its greatest speed is at the equilibrium point.

When the spring reaches its maximum compression, the object stops for an instant. At this point, its speed equals zero. The spring then expands until the object returns to its initial position, with the spring fully extended. Again, the object stops for an instant, and its speed is zero.

In the paragraphs above, we discussed the motion in terms of speed, not velocity, so we could ignore the sign and focus on how fast the object moves. The object’s velocity will be both positive and negative as it moves back and forth. You see this alternating pattern of positive and negative velocities in the graph in Equation 1.

When the displacement is at an extreme, the velocity is zero, and vice-versa. One way to state the relationship between the displacement and velocity functions is to say they are $\pi/2$ radians ($90^\circ$) out of phase. An equivalent way to express this without a phase constant is to use a cosine function for displacement and a sine function for velocity, and this is what we do. This relationship can also be derived using calculus. In Equation 1, you see both a velocity graph and the function for velocity.

The second equation shown in Equation 1 states that the maximum speed $v_{\text{max}}$ is the amplitude of the displacement function times the angular frequency. To understand the source of this equation, recall that the maximum magnitude of the sine function is one. When the sine has a value of $-1$ in the velocity equation, the velocity reaches its maximum value of $A\omega$. 

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Velocity in SHM

\[ v(t) = -A\omega \sin (\omega t + \varphi) \]

\[ v_{\text{max}} = A\omega \]

- \( v \) = velocity
- \( A \) = amplitude
- \( \omega \) = angular frequency
- \( t \) = time
- \( \varphi \) = phase constant

What is the velocity at \( t = 4.0 \) seconds?

\[ v(t) = -A\omega \sin (\omega t + \varphi) \]

\[ v(4.0 \text{ s}) = (-2\pi) \sin \left( \frac{2\pi \text{ rad}}{3} \right) \text{ m/s} \]

\[ v(4.0 \text{ s}) = -(6.28)(0.866) \text{ m/s} \]

\[ v(4.0 \text{ s}) = -5.4 \text{ m/s} \]

A particle vibrates in simple harmonic motion with a frequency of \( 2.60 \times 10^8 \) Hz and an amplitude of \( 1.70 \times 10^{-8} \) m. What is its maximum speed?

Answer:

\[ v_{\text{max}} = \text{m/s} \]
For SHM to occur, the net force on an object has to be proportional and opposite in sign to its displacement. Again, we use the example of a mass attached to a spring on a friction-free surface, like an air hockey table.

With a spring like the one shown in Concept 1 to the right, Hooke’s law \((F = -kx)\) states the relationship between net force and displacement from equilibrium. This equation for force enables you to determine where the acceleration is the greatest, and where it equals zero. The magnitudes of the force and the acceleration are greatest at the extremes of the motion, where \(x\) itself is the greatest. This is the point where the object is changing direction. Conversely, \(x = 0\) at the equilibrium point, so \(F = 0\) and the object is not accelerating there.

The first equation shown in Equation 1 enables you to calculate the acceleration of an object in SHM as a function of time. This equation can be simplified by noting that the amplitude times the cosine function, the rightmost term in the equation, is the function for the object’s displacement, \(x(t)\). We replace the terms \(A \cos \omega t\) by \(x(t)\) to derive the second equation, which relates the acceleration directly to the object’s displacement. This equation says that the acceleration at a particular time equals the negative of the angular frequency squared times the object’s displacement at that time.

Finally, the third equation reveals that the maximum acceleration of the object is the amplitude times the square of the angular frequency. This equation is a consequence of the first equation.

\[
\begin{align*}
\text{Acceleration in SHM} \\
\text{Proportional to force} & \cdot \text{Zero at equilibrium} & \cdot \text{Maximum at extremes}
\end{align*}
\]

\[
\begin{align*}
\text{equation 1} \\
x(t) &= A \cos(\omega t + \phi) \\
a(t) &= -A\omega^2 \cos(\omega t + \phi)
\end{align*}
\]

\[
\begin{align*}
\text{Acceleration in SHM} \\
a(t) &= -A\omega^2 \cos(\omega t + \phi) \\
a(t) &= -\omega^2 x(t) \\
\alpha_{\text{max}} &= A\omega^2 \\
a = \text{acceleration}, A = \text{amplitude} \\
\omega = \text{angular velocity} \\
x = \text{displacement}, t = \text{time} \\
\phi = \text{phase constant}
\end{align*}
\]

\[
\begin{align*}
\text{example 1} \\
x(t) &= 2.5 \cos(\omega t) \\
\alpha(1.6s) &= -2.5m[s^{-1}]^2 \cos(\frac{\pi}{s}\cdot 1.6s) \\
\alpha(1.6s) &= -7.6 \text{ m/s}^2
\end{align*}
\]

What is the acceleration at \(t = 1.6\) seconds?

\[
\begin{align*}
a(t) &= -A\omega^2 \cos(\omega t + \phi) \\
a(1.6s) &= -2.5m[s^{-1}]^2 \cos(\frac{\pi}{s}\cdot 1.6s) \\
a(1.6s) &= -7.6 \text{ m/s}^2
\end{align*}
\]
15.12 - Sample problem: calculating period from acceleration

The ball moves in SHM, and its acceleration is defined by the equation shown. What is the period of the ball's motion?

\[ a(t) = -28.0 x(t) \]

**Variables**

<table>
<thead>
<tr>
<th>acceleration</th>
<th>[ a(t) = -28.0 \cdot x(t) ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>angular frequency</td>
<td>[ \omega ]</td>
</tr>
<tr>
<td>period</td>
<td>[ T ]</td>
</tr>
</tbody>
</table>

**What is the strategy?**

1. Use the equation relating acceleration to angular frequency and displacement, and the given equation, to calculate the angular frequency.
2. Calculate the period from the angular frequency.

**Physics principles and equations**

The acceleration of an object in SHM is given by

\[ a(t) = -\omega^2 x(t) \]

Period and angular frequency are related by the equation

\[ \omega = \frac{2\pi}{T} \]

**Step-by-step solution**

There are two equations for acceleration: the one given in the problem, and one that relates acceleration to angular frequency and displacement. We use these equations to calculate the angular frequency.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>[ a(t) = -\omega^2 x(t) ] acceleration equation</td>
</tr>
<tr>
<td>2.</td>
<td>[ a(t) = -(28.0) x(t) ] equation stated in problem</td>
</tr>
<tr>
<td>3.</td>
<td>[ -\omega^2 x(t) = -(28.0) x(t) ] set right sides of steps 1, 2 equal</td>
</tr>
<tr>
<td>4.</td>
<td>[ \omega^2 = 28.0 ] divide by (-x(t))</td>
</tr>
<tr>
<td>5.</td>
<td>[ \omega = 5.29 \text{ rad/s} ] square root</td>
</tr>
</tbody>
</table>

Now we calculate the period from the angular acceleration.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.</td>
<td>[ \omega = \frac{2\pi}{T} ] equation for period</td>
</tr>
<tr>
<td>7.</td>
<td>[ T = \frac{2\pi}{\omega} ] solve for ( T )</td>
</tr>
<tr>
<td>8.</td>
<td>[ T = \frac{2\pi}{(5.29 \text{ rad/s})} ] enter angular acceleration</td>
</tr>
<tr>
<td>9.</td>
<td>[ T = 1.19 \text{ s} ] evaluate</td>
</tr>
</tbody>
</table>

15.13 - Summary of simple harmonic motion

You have now been exposed (subjected?) to three trigonometric functions, and three graphs, for the displacement, velocity and acceleration of an object in simple harmonic motion. All of these equations use the sine or cosine function; these are called sinusoidal functions. We show the equations for these properties of motion in Equation 1.

You may find it useful to see the three graphs of these functions together, as shown in Concept 1. Remember that the vertical axes represent different units. Going from the top graph to the bottom one, the axes are meters, m/s, and m/s².
The graphs illustrate the relationship of displacement, velocity and acceleration. You can “see” some of the nature of simple harmonic motion in these graphs. For example, when the particle is at one of its endpoints, displacement has the greatest magnitude, and velocity is zero as the particle momentarily pauses and changes the sign of its velocity. When the displacement has its greatest positive value, the acceleration has its most negative value, and vice versa.

Summary of SHM
Sinusoidal functions describe displacement, velocity, acceleration

\[
x(t) = A \cos (\omega t + \phi)
\]
\[
v(t) = -A\omega \sin (\omega t + \phi)
\]
\[
a(t) = -A\omega^2 \cos (\omega t + \phi)
\]

\(x = \) displacement, \(v = \) velocity
\(a = \) acceleration, \(A = \) amplitude
\(\omega = \) angular frequency
\(t = \) time, \(\phi = \) phase constant

15.14 - Simple harmonic motion and uniform circular motion

Galileo was the first to make observations of the moons of Jupiter. He used a telescope, at that time a recent invention, to discover four of the planet’s moons. He noted that the moons move in a pattern that could be best explained if the objects he observed were moving in circles around Jupiter. Using today’s vocabulary, we would say that the pattern of motion he observed could be described using the concepts of simple harmonic motion. This is not to say the moons move in SHM, because they do not. But they appear to be doing so because an object moving in uniform circular motion viewed edge-on from afar appears to be moving in SHM.

To help you understand Galileo’s insight, we will explain in more depth what he observed. The moons’ orbits are roughly circular. From his perspective on Earth, Galileo could only see the moons’ lateral motion, their motion from left to right and right to left. At the moons’ great distance, their motion toward and away from the Earth was not perceptible. The moons seem to repeatedly move back and forth along a straight line, as objects in SHM do.

When an object moving in uniform circular motion, such as a moon of Jupiter, is observed “edge on,” does the repeated motion actually conform to the equations for SHM? In Concept 1 we show a ball moving up and down in circular motion. The graph shows its vertical displacement and, as you can see, the graph is sinusoidal, like those that represent displacement in SHM.

In Equation 1, you can see why we perceive edge-on uniform circular motion as SHM. Consider the \(x\) displacement of the particle moving in uniform circular motion. The \(x\) displacement equals the radius (which we will call \(A\) here) times the cosine of the angle \(\theta\). The angle \(\theta\) is the angular displacement. As you may recall from your studies of rotational motion, angular displacement equals the product of angular velocity and time, or \(\omega t\). As the equation to the right reflects, the function for the \(x\) displacement is the same function as the one used for calculating the displacement of an object in SHM.

The relationship between uniform circular motion and SHM can also be confirmed qualitatively. At the perceived endpoints of its circular path, a moon of Jupiter would be moving directly away from, or toward, Galileo. In other words, its tangential velocity is directed toward or away from the Earth at these points. Here its velocity should seem to go to zero just as in SHM, and this is what Galileo observed.

Conversely, at the midpoint of its motion on the Earth side of Jupiter, a moon would seem to be moving the fastest because of the orientation of its tangential velocity. And
in SHM, the greatest speed is observed at the midpoint of motion.

\[ \omega = \text{angular velocity} \]
\[ t = \text{time} \]

15.15 - Period, spring constant, and mass

In Equation 1, you see two equations that relate the physical properties of a mass-spring system moving in SHM to its angular frequency and period. These reflect how the physical properties of this system determine its motion. These equations are applicable when the mass of the spring is negligible compared to the object attached to it.

The first equation states that the angular frequency equals the square root of the spring constant divided by the mass. You can think of this equation in terms of Hooke’s law and Newton’s second law. Hooke’s law states that for a given displacement, the greater the spring constant, the greater the force. This means there will be greater acceleration and the mass will both reach its peak velocity and return to zero velocity more quickly. This gives you reason to think the frequency of motion increases with the spring constant.

In contrast, by Newton’s second law acceleration is inversely proportional to mass. As the mass increases the acceleration decreases, so angular frequency decreases with mass.

The second equation, an equation for period, can be derived from the first equation using the relationship \( \omega = \frac{2\pi}{T} \). Since the period is inversely proportional to the angular frequency, the period increases as mass increases and decreases as the spring constant increases.

Notice that the angular frequency and the period depend only on the mass and spring constant, not on the amplitude. When the amplitude increases, the mass has to travel farther during a cycle. However, increasing the amplitude also increases the maximum force applied to the mass, which increases its maximum acceleration and results in a greater average speed. The increased average speed means the mass completes the cycle in the same amount of time.

To derive the first equation, we use Newton’s second law and Hooke’s law. We also use an equation discussed earlier to calculate the acceleration as a function of the angular frequency. Then using algebra we can solve for the angular frequency.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( F = ma )</td>
</tr>
<tr>
<td>2.</td>
<td>( a = -\omega^2 x )</td>
</tr>
<tr>
<td>3.</td>
<td>( F = -m\omega^2 x )</td>
</tr>
<tr>
<td>4.</td>
<td>( F = -kx )</td>
</tr>
<tr>
<td>5.</td>
<td>( -m\omega^2 x = -kx )</td>
</tr>
<tr>
<td>6.</td>
<td>( \omega = \sqrt{\frac{k}{m}} )</td>
</tr>
</tbody>
</table>

In Example 1, we calculate the period of a mass on a spring.

On Skylab, astronaut Alan Bean used the period of a spring-driven chair to measure his mass.

Angular frequency and period

\[ \omega = \sqrt{\frac{k}{m}} \]
\[ T = \frac{2\pi}{\sqrt{\frac{m}{k}}} \]

\( \omega = \text{angular frequency} \)
\( T = \text{period} \)
\( k = \text{spring constant} \)
\( m = \text{mass} \)

Example 1

What is the period of the puck’s motion?

\[ T = 2\pi \sqrt{\frac{m}{k}} \]

\( k = 29 \text{ N/m} \)
\( m = 0.45 \text{ kg} \)
15.16 - Interactive problem: match the curve again

In the simulation on the right, you control parameters for a puck on a spring moving in simple harmonic motion. With the right settings, the puck will create a graph that matches the one shown on the paper.

You can determine the amplitude by examining the graph. The spring constant is 10.0 N/m. The desired period can be determined by examining the graph, but you do not set the period directly. Instead, you set the mass of the puck, which determines the period. Use the spring constant and the period to calculate what the mass should be. Enter the amplitude to the nearest 0.01 m and the mass to the nearest 0.01 kg and press GO. A gauge shows the actual period so that you can confirm your calculations. If your graph does not match the target graph, press RESET to try again.

You should review the section on springs and period if you have trouble solving this problem.

15.17 - Sample problem: isolation platform

Laboratories isolate sensitive equipment from floor vibrations by using a platform resting on springs, as you see above. The springs are chosen so that the frequency of vibration of the platform plus the equipment is below 2 Hz. The frequency of vibrations from the floor is typically in the 5–30 Hz range. This difference in frequencies allows the isolation platform to "filter out" the floor vibrations before they reach the equipment. In the platform above, each of the four identical springs supports one-fourth of the total mass.

Variables

<table>
<thead>
<tr>
<th>Variables</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>mass of platform</td>
<td>( m_1 = 24.0 \text{ kg} )</td>
</tr>
<tr>
<td>mass of equipment</td>
<td>( m_2 = 5.50 \text{ kg} )</td>
</tr>
<tr>
<td>spring constant for a spring</td>
<td>( k = 425 \text{ N/m} )</td>
</tr>
<tr>
<td>angular frequency of system</td>
<td>( \omega )</td>
</tr>
<tr>
<td>frequency of a spring</td>
<td>( f )</td>
</tr>
</tbody>
</table>

What is the strategy?

Since the springs are identical, and each supports an equal fraction of the total mass (and weight), they have the same frequency. We calculate the frequency of one spring as follows.

1. Calculate the angular frequency of a system consisting of one spring, supporting one-fourth of the total mass.
2. Calculate the frequency from the angular frequency.

Physics principles and equations

The angular frequency of a mass on a spring is

\[
\omega = \sqrt{\frac{k}{m}}
\]

Frequency and angular frequency are related by the equation

\[
\omega = 2\pi f
\]

Step-by-step solution
We start by considering a system consisting of one spring supporting one-fourth the total mass.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \omega = \frac{\sqrt{k}}{\sqrt{m}} )</td>
<td>equation for angular frequency of mass on spring</td>
</tr>
<tr>
<td>2. ( \omega = \frac{k}{\sqrt{(m_1 + m_2)/4}} )</td>
<td>mass supported by one spring</td>
</tr>
<tr>
<td>3. ( \omega = \frac{425 \text{ N/m}}{\sqrt{(24.0 \text{ kg}) + (5.50 \text{ kg})/4}} )</td>
<td>enter values</td>
</tr>
<tr>
<td>4. ( \omega = 7.59 \text{ rad/s} )</td>
<td>evaluate</td>
</tr>
</tbody>
</table>

Now that we have the angular frequency of the system, we can calculate the frequency.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. ( \omega = 2\pi f )</td>
<td>relationship of angular frequency and frequency</td>
</tr>
<tr>
<td>6. ( f = \omega/2\pi )</td>
<td>solve for ( f )</td>
</tr>
<tr>
<td>7. ( f = (7.59 \text{ rad/s})/2\pi )</td>
<td>enter angular frequency</td>
</tr>
<tr>
<td>8. ( f = 1.21 \text{ Hz} )</td>
<td>evaluate</td>
</tr>
</tbody>
</table>

The frequency of 1.21 Hz is in the desirable range for an isolation platform.

15.18 - Work and the potential energy of a spring

We will once again use a spring as our tool for analyzing simple harmonic motion. Springs can store potential energy. The energy stored by a spring is called elastic potential energy.

When the spring shown in Concept 1 is extended, it has elastic potential energy. When the hand releases the mass at the end of the spring, potential energy will be converted into kinetic energy as the spring and mass move.

It takes work to stretch or compress a spring. The work done on the spring-mass system will equal the change in total energy of the system. It is useful to consider the system when it has only potential energy, for instance, when we pull back on the mass from its rest (equilibrium) position and hold it still. We define the potential energy of the system to be zero when the mass is at its equilibrium position, so the work we do on the system as we stretch the spring and then hold it equals the potential energy stored in the stretched spring.

The formula in Equation 1 states that the potential energy equals one-half the spring constant \( k \) times the square of the displacement \( x \). Although we use the example of the spring and the spring constant, this expression can be used to calculate the potential energy of any system with a restoring force proportional to displacement, with the appropriate restoring force constant \( k \).

We can graphically derive the value for potential energy shown in the equation. In Equation 1, you see a graph that shows how the displacement, force and work are related.

1. The potential energy stored by the spring when it is stretched equals the work done on the system by the pulling hand. At any displacement \( x \), the hand exerts a force \( kx \), opposing the spring force. The linear graph shows this force as a function of \( x \).
2. When force varies with displacement as it does here, the work done equals the area under the graph. The region under the graph is a right triangle whose area equals one-half the product of its base times its height. The base of the triangle is the displacement \( x \) and the height is the force \( kx \). The area of the triangle, then, is \( \frac{1}{2} kx^2 \), which equals the work done, and the \( PE \) as well.

\[ PE = \frac{1}{2} kx^2 \]

\( PE \) = potential energy
\( k \) = spring constant
How much potential energy is stored in the spring?

\[ PE = \frac{1}{2} kx^2 \]

\[ PE = \frac{1}{2}(320 \text{ N/m})(0.12 \text{ m})^2 \]

\[ PE = 2.3 \text{ J} \]

15.19 - Total energy

When an object on a spring moves in simple harmonic motion, the system’s energy changes back and forth from elastic potential energy to kinetic energy. Assuming no non-conservative forces like friction are present, the law of conservation of energy dictates that the total energy of the system remains constant over time.

In Concept 1, you see a graph of the displacement of the puck as it oscillates in SHM. There are also three energy gauges for the mass/spring system. As the puck moves and the spring expands and contracts, the values displayed in the \( PE \) and \( KE \) gauges will change, but their sum, the total energy (\( TE \)), remains constant.

At its maximum displacement, the puck has zero kinetic energy because it has zero velocity. All of the energy of the system is elastic potential energy in the spring. This is the situation shown in Concept 1. At the equilibrium position, where the puck is moving the fastest and the spring is relaxed, all of the system’s energy is kinetic energy. We have defined the equilibrium position as the point of zero potential energy.

In Equation 1, you see equations for the two kinds of energy in a system consisting of a mass \( m \) and a spring with constant \( k \). The kinetic energy equals \( \frac{1}{2} mv^2 \) as always, but here we express its velocity using the sinusoidal velocity expression for an object in SHM.

As was shown in the prior section, the potential energy arising from the restoring force of an object in simple harmonic motion is \( \frac{1}{2} kx^2 \), where \( x \) is the displacement of the object. In the second equation in Equation 1, we write this using the sinusoidal expression for the displacement of an object in SHM.

We combine the graphs for \( KE \) and \( PE \) in Equation 2. The values for the graphs sum to a constant at each point. The total energy of the system does not change, even as its components do.

To determine an equation for total energy, consider the puck when it is at its farthest point from equilibrium, where \( x = A \). At this point, the puck has zero velocity, so there is no kinetic energy. The total energy is solely the potential energy, which equals \( \frac{1}{2} kA^2 \).

As the puck begins to move, its \( KE \) increases as its \( PE \) decreases, but the total energy remains the same. We show this as an equation in Equation 2 on the right.

**Total energy**

Total energy is constant (\( KE + PE \))

- At equilibrium: all \( KE \)
- At maximum displacement: all \( PE \)

**Kinetic, potential energy**

\[ KE = \frac{1}{2} mA^2 \omega^2 \sin^2 \omega t \]

\[ PE = \frac{1}{2} kA^2 \cos^2 \omega t \]

- \( m \) = mass
- \( A \) = amplitude
- \( \omega \) = angular frequency
- \( t \) = time
- \( k \) = restoring force constant
When the friction-free iron is released, what will its KE be at its equilibrium point?
Total energy at equilibrium is all KE
KE at equilibrium = initial PE
KE = \( \frac{1}{2} kA^2 \)
KE = \( \frac{1}{2} (38 \text{ N/m})(0.30 \text{ m})^2 \)
KE = 1.7 J

A disk mounted on a massless spring oscillates in simple harmonic motion. The disk has a mass of 0.580 kg and moves with an amplitude of 0.150 m. The total energy of the system is 1.30 J. What is the system's period?

Answer:
\[ T = \text{seconds} \]
In many prior examples, we showed a mass attached to a spring on a frictionless horizontal surface. The only force we needed to consider was the restoring force from the spring. Here, since the mass is attached vertically to a spring, we must account for the force of gravity as well.

The two forces acting on the block are the downward force of gravity and the upward force of the spring (we ignore other forces like air resistance). The distance the block falls before stopping can be calculated using the conservation of energy. In this case, we need to consider gravitational potential energy as well as the potential energy in the spring. We ignore the mass of the spring itself.

**Draw a diagram**

We let \( h \) represent the distance the block falls. We consider the gravitational potential energy of the block to be zero at the bottom of the block’s motion.

**Variables**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>mass of block</td>
<td>( m = 0.230 \text{ kg} )</td>
</tr>
<tr>
<td>spring constant</td>
<td>( k = 37.0 \text{ N/m} )</td>
</tr>
<tr>
<td>initial block height</td>
<td>( h )</td>
</tr>
<tr>
<td>final block height</td>
<td>0 m</td>
</tr>
<tr>
<td>initial total energy of block</td>
<td>( TE_i )</td>
</tr>
<tr>
<td>final total energy of block</td>
<td>( TE_f )</td>
</tr>
</tbody>
</table>

**What is the strategy?**

1. Use the conservation of energy to state that the total energy before the block is released is the same as the total energy at the turnaround point.
2. The block is not moving initially, and is (momentarily) not moving when it changes direction. Its kinetic energy is zero at both of these positions. Use this fact to simplify the equation.
3. Solve for the distance \( h \).

**Physics principles and equations**

Conservation of energy

\[ TE_i = TE_f \]

Total energy of an object in simple harmonic motion

\[ TE = PE + KE \]

Gravitational potential energy

\[ PE = mgh \]

Elastic potential energy from a spring

\[ PE = \frac{1}{2}kx^2 \]
Step-by-step solution

Since the block has zero velocity at both the top and bottom positions of its motion, its initial and final kinetic energies are both zero. The total energy, in both the initial and final states, equals the potential energy. The initial potential energy of the block is all in the form of gravitational potential energy since the spring is unstretched. The potential energy as the block changes direction at the bottom of its motion is all due to the elastic potential energy in the spring, because we set the gravitational PE to zero there.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $TE_i = TE_f$</td>
<td>conservation of energy</td>
</tr>
<tr>
<td>2. $PE_i + KE_i = PE_f + KE_f$</td>
<td>total energy of object</td>
</tr>
<tr>
<td>3. $PE_i = PE_f$</td>
<td>kinetic energies are zero</td>
</tr>
<tr>
<td>4. $mgh = \frac{1}{2}kh^2$</td>
<td>enter expressions for potential energies</td>
</tr>
</tbody>
</table>

Now we have an equation where the only unknown value is the one we want to calculate.

<table>
<thead>
<tr>
<th>Step</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>5. $h = \frac{2mg}{k}$</td>
<td>solve for $h$</td>
</tr>
<tr>
<td>6. $h = \frac{2(0.230 \text{ kg})(9.80 \text{ m/s}^2)}{37.0 \text{ N/m}}$</td>
<td>enter values</td>
</tr>
<tr>
<td>7. $h = 0.122 \text{ m}$</td>
<td>evaluate</td>
</tr>
</tbody>
</table>

15.22 - A torsional pendulum

The torsional pendulum shown in Concept 1 is another device that exhibits simple harmonic motion. A torsional pendulum consists of a mass suspended at the end of a stiff rod, wire or spring. It does not swing back and forth. Instead, the mass at the bottom is initially rotated by an external torque away from its equilibrium position. The elasticity of the rod supplies a restoring torque, causing the mass to rotate back to the equilibrium position and beyond. The mass rotates in an angular version of simple harmonic motion.

Earlier, we stated that for SHM to occur, the force must be proportional to displacement. Since torsional pendulums rotate, we must use angular concepts to analyze them. With a torsional pendulum, a restoring torque, not a force, acts to return the system to its equilibrium position. The restoring torque is proportional to angular displacement, just as a restoring force is proportional to (linear) displacement. The moment of inertia of the system takes on the role that mass plays in linear SHM.

The same analysis that applies to linear displacement, velocity and acceleration applies equally well to angular displacement, angular velocity and angular acceleration. In Equation 1, you see an equation that states the nature of the restoring torque. It equals the negative of the product of the torsion constant and the angular displacement.

The formula in Equation 2 calculates the period of the pendulum. When the period and the torsion constant are known, the moment of inertia can be calculated, as shown in Example 1. This makes torsional pendulums useful tools for experimentally determining the moments of inertia of complex objects.

Restoring torque

$$\tau = -\kappa \theta$$

$\tau = \text{torque}$
$\kappa = \text{torsion constant}$
$\theta = \text{angular displacement}$
Units for $\kappa$: N·m/rad

**Period**

$$T = 2\pi \sqrt{\frac{I}{\kappa}}$$

$T = \text{period}$
$I = \text{moment of inertia}$
$\kappa = \text{torsion constant}$

---

**Example 1**

The torsional pendulum has a period of 3.0 s. What is its moment of inertia?

$$T = 2\pi \sqrt{\frac{I}{\kappa}}$$

$$I = \frac{\kappa T^2}{4\pi^2}$$

$$I = (0.088 \text{ N·m/rad})(3.0 \text{ s})^2/4\pi^2$$

$I = 0.020 \text{ kg·m}^2$

---

**15.23 - A simple pendulum**

Old-fashioned “grandfather” clocks, like the one you see in Concept 1, rely on the regular motion of their pendulums to keep time. A typical pendulum is constructed with a heavy weight called a “bob” attached to a long, thin rod. The bob swings back and forth at the end of the rod in a regular motion.

We approximate such a system as a simple pendulum. In a simple pendulum, the bob is assumed to be concentrated at a single point located at the very end of a cable, and the cable itself is treated as having no mass. The system is assumed to have no friction and to experience no air resistance. When such a pendulum swings back and forth with a small amplitude, its angular displacement closely approximates simple harmonic motion. This means the period does not vary much with the pendulum’s amplitude. This regularity of period is what makes pendulums useful in clocks.

For SHM to occur, the restoring force or torque needs to vary linearly with displacement. In the case of a pendulum, the motion is rotational, so the torque must be linearly proportional to the angular displacement.

In Equation 1, you see a free-body diagram of the forces on the pendulum bob. The tension in the cable exerts no torque on the pendulum since it passes through its center of rotation, so the weight $mg$ of the bob exerts the only torque. The lever arm of this weight equals the length $L$ of the cable times $\sin \theta$. For small angles, the angle expressed in radians is a very close approximation of the sine of the angle. (The error is less than 1% for angles less than $14^\circ$.) This means that the resulting torque is roughly proportional to the angular displacement, and the condition for SHM is approximated, with a torsion constant of $mgL$.

In Equation 2, you see the equation for the period of a simple pendulum. When the angular amplitude is small and the approximation mentioned above is used, the period depends solely on the length of the cable and the acceleration of gravity.

A pendulum can be an effective tool for measuring the acceleration caused by gravity using the equation just mentioned. The length $L$ of the cable is measured and the pendulum is set swinging with a small amplitude. The period $T$ is then measured. The value of $g$ can be calculated using the rearranged equation $g = 4L(\pi/T)^2$. 

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Restoring torque

\[ \tau = -mgL \sin \theta \approx -mgL \theta \]

\( \tau \) = torque
\( m \) = mass
\( g \) = acceleration of gravity
\( L \) = length of pendulum
\( \theta \) = angular displacement

Period

\[ T = 2\pi \sqrt{\frac{L}{g}} \]

\( T \) = period
\( L \) = length of pendulum
\( g \) = acceleration of gravity

What is the period of this pendulum?

\[ T = 2\pi \sqrt{\frac{1.3 \text{ m}}{9.80 \text{ m/s}^2}} \]

\[ T = 2.3 \text{ s} \]

On the right is a simulation of a simple pendulum: a bob at the end of a string. You can control the length of the string, and in doing so change the period of the pendulum. Your goal is to set the length so that the period is 2.20 seconds. As the pendulum swings, you will see a graph reflecting the angular displacement of the bob.

Calculate and set the value for the string length to the nearest 0.05 m using the dial, then use your mouse to drag the bob to one side and release it to start the pendulum swinging. There may not be enough room in the window to show the entire length of the string, but we will show the motion of the bob and the resulting period. If you do not set the length correctly, press RESET to try again. Refer to the section on simple
pendulums if you do not remember the equation for the period. You may want to double-check your work by creating an actual pendulum with a string of the correct length. You can time it: Ten cycles of its motion should take about 22 seconds.

For small angles, the angular displacement of a pendulum approximates simple harmonic motion and the graph looks sinusoidal. Try smaller and larger angles and observe the graphs. How sinusoidal do they look to you? (In the simulation, decreasing the string length makes it easier to create large angular displacements.) You can check the box labeled "SHM" to draw a sinusoidal graph of SHM motion in black underneath your red graph. The black graph shows simple harmonic motion for the amplitude you choose and the period calculated by the pendulum equation. If the amplitude is small, you might not see the black graph, because the two graphs match so closely.

15.25 - Period of a physical pendulum

Not all pendulums are simple. A physical pendulum is a rigid extended object (not a point mass) pivoting around a point. In Concept 1, you see a violin acting as a physical pendulum.

The equation for the period of a physical pendulum is shown in Equation 1. The distance \( h \) is the distance from the pivot point to the center of mass of the object. As always, the moment of inertia must be calculated about the pivot point.

The example problem shows how to calculate the period of a meter stick used as a pendulum in the Earth’s gravitational field. The period is 1.6 seconds. With the use of a meter stick, this is a result you can verify for yourself. If the stick has a hole close to one end, put an unbent paper clip through the hole (otherwise, pinch the end very loosely between your fingers), and set the stick swinging. Ten swings should take approximately 16 seconds.

\[
T = 2\pi \sqrt{\frac{I}{mgh}}
\]

\( T \) = period
\( I \) = moment of inertia
\( m \) = mass, \( g \) = acceleration of gravity
\( h \) = distance from pivot to center of mass
What is the period of the swinging meter stick?

\[ T = 2\pi \sqrt{\frac{I}{mgL}} \]

\[ I = \frac{1}{3} \times \text{mass} \times \text{length}^2 \]

\[ T = 2\pi \sqrt{\frac{1/3 (0.11 \text{ kg})(1.0 \text{ m})^2}{(0.11 \text{ kg}) (9.80 \text{ m/s}^2) (0.50 \text{ m})}} \]

\[ T = 2\pi \sqrt{0.068} = 1.6 \text{ s} \]

15.26 - Sample problem: meter-stick pendulum

A meter stick swings as a pendulum, with an axis of rotation 0.250 meters from the end. What is its period?

Since the meter stick is a physical pendulum, its period depends on its moment of inertia. The meter stick is assumed to be uniform. In order to calculate its moment of inertia, we will treat it as a thin rod and use the parallel axis theorem.

Notice that you are not given the mass of the meter stick. It turns out it is not needed to solve the problem.

Diagram

We show the center of mass, labeled CM, and the distance from the center of mass to the axis of rotation, labeled \( h \). The distance \( h \) will be needed to calculate the moment of inertia.

Variables

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>length of stick</td>
<td>( L = 1.00 \text{ m} )</td>
</tr>
<tr>
<td>distance from axis to CM</td>
<td>( h = 0.250 \text{ m} )</td>
</tr>
<tr>
<td>mass</td>
<td>( m )</td>
</tr>
<tr>
<td>moment of inertia</td>
<td>( I )</td>
</tr>
</tbody>
</table>
What is the strategy?

1. Use the parallel axis theorem to calculate the moment of inertia of the meter stick for the axis of rotation. This expression will involve the mass.
2. Use the moment of inertia to calculate the period. The mass will cancel out.

Physics principles and equations

The period of the physical pendulum depends on its moment of inertia, mass and the distance from the pivot to the center of mass.

\[ T = 2\pi \sqrt{\frac{l}{mg\theta}} \]

The parallel axis theorem calculates the moment of inertia for an axis of rotation parallel to one passing through the center of mass.

\[ I = I_{CM} + mh^2 \]

For a thin rod, the moment of inertia for an axis of rotation passing through the center of mass is given by

\[ I_{CM} = \frac{1}{12} mL^2 \]

Step-by-step solution

We first calculate the moment of inertia of the meter stick about the indicated axis, using the parallel axis theorem. We do not know the mass, so we leave that variable in for now.

\[
\begin{align*}
1. & \quad I = I_{CM} + mh^2 \\
2. & \quad I = \frac{1}{12} mL^2 + mh^2 \\
3. & \quad I = \frac{1}{12} (m \text{ kg})(1.00 \text{ m})^2 + (m \text{ kg})(0.250 \text{ m})^2 \\
4. & \quad I = (0.146)m \text{ kg} \cdot \text{m}^2 \\
\end{align*}
\]

Now we use the moment of inertia to find the period.

\[
\begin{align*}
5. & \quad T = 2\pi \sqrt{\frac{l}{mg\theta}} \\
6. & \quad T = 2\pi \sqrt{\frac{(0.146)m \text{ kg} \cdot \text{m}^2}{(m \text{ kg})(9.80 \text{ m/s}^2)(0.250 \text{ m})}} \\
7. & \quad T = 2\pi \sqrt{\frac{0.146 \text{ m}^2}{(9.80 \text{ m/s}^2)(0.250 \text{ m})}} \\
8. & \quad T = 1.53 \text{ s} \\
\end{align*}
\]

In a previous section, we found that the period for a meter stick rotating at its end was a slightly longer 1.6 seconds.

15.27 - Damped oscillations

We have considered many types of oscillations, and up until now assumed the periodic motion continued without change. But most real-world oscillations are damped, which means they are subject to forces like friction that cause the amplitude of the motion to decrease over time.

Mountain bike shock absorbers provide an excellent demonstration of damped oscillations. A shock absorber often combines a spring with a sealed container of fluid. Shock absorbers lessen the jolts of a bumpy trail.

To explain this in more detail, let's consider what happens when a bike equipped with such a shock absorber hits a bump. The force from the bump compresses the spring, with the result that less of the force from the bump passes to the rest of the bike (and the rider).

The spring then supplies a restoring force. In the absence of any other force, the rider and bike would in principle then move forever in simple harmonic motion. However,
inside a shock absorber, the spring moves a piston in a sealed cylinder of fluid. The fluid supplies what is called a *damping force*.

In Concepts 1 and 2, you see a diagram of this system. The fluid (typically oil) provides a force that opposes the motion of the piston. The damping force always opposes (resists) the motion of an object, which means sometimes it acts in the same direction as the restoring force (when the object moves away from equilibrium), and sometimes in the opposite direction (when the object moves toward equilibrium). At all times, however, it is opposing the motion.

Instead of moving in SHM, the system moves back to its equilibrium point and stops, or it may oscillate a few times with smaller and smaller amplitude before resting at its equilibrium point. The fluid “dampens” the motion, reducing the amplitude of the oscillations. The result is a relatively fast yet smooth return to the equilibrium position.

The resistive force of the fluid in a system like this is often proportional to the velocity, and opposite in direction. In Equation 1, you see the equation for the damping force. It equals the negative of \( b \) (the damping coefficient) times the velocity. (You may note that this is similar to the formula for air resistance, where the drag force depends on the square of the velocity.) The negative sign indicates that the damping force opposes the motion that causes it.

In Equation 2, you also see the equation for the net force \( F_N \). The net force is the sum of the restoring forces and the damping force. (If you look at the equation, it may seem that two negatives combine to make a larger number, but the sign of the velocity is the opposite of the displacement as the system moves toward equilibrium.)

The graph in Equation 3 illustrates three types of damping. The blue line represents a critically damped system. The damping force is such that the system returns to equilibrium as quickly as possible and stops at that point.

The green line represents a system that is overdamped. The damping force is greater than the minimum needed to prevent oscillations. The system returns to equilibrium without oscillating, but it takes longer to do so than a critically damped system.

The red line is a system that is underdamped. It oscillates about the equilibrium point, with ever diminishing amplitude.

With certain shock absorbers, the system can be adjusted, which means that the damping coefficient can be tuned based on rider preferences. Beginners often prefer an underdamped system. The bike bounces a bit but there is less of a “jolt” because the shock absorber acts more slowly. Advanced riders sometimes prefer a critically damped or overdamped “harder” ride, trading off a less smooth ride in exchange for regaining control of the bicycle more quickly.

---

**Damping force**

Opposes motion

Often proportional to velocity

**Equation 1**

\[
F_d = -bv
\]

\( F_d \) = damping force

\( b \) = damping coefficient

\( v \) = velocity

**Equation 2**

Net force

\[
\Sigma F = -kx - bv
\]

\( \Sigma F \) = net force

\( k \) = spring constant

\( x \) = displacement

\( b \) = damping coefficient

\( v \) = velocity

**Equation 3**

Types of damped harmonic motion

Critically damped: blue line

Overdamped: green line

Underdamped: red line

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Forced oscillation: A periodic external force acts on an object, increasing the amplitude of its motion.

External forces can dampen, or reduce, the amplitude of harmonic motion. For instance, in a mass-spring system, friction reduces the amplitude of the mass’s motion over time. External forces can also maintain or increase the amplitude of an oscillation, counteracting damping forces.

Consider a child on a swing. Friction and air resistance are damping forces that reduce the amplitude of the motion. On the other hand, an external force like a person pushing, as you see in Concept 1, can increase the amplitude. When an external force increases the amplitude, forced oscillation occurs.

An external force that acts to increase the amplitude of oscillations is called a driving force. The driving force oscillates at a frequency called the driving frequency.

The natural frequency of a system is the frequency at which it will oscillate in the absence of any external force. Systems have natural frequencies based on their structure. The closer the driving frequency is to the natural frequency, the more efficiently the driving force transfers energy to the system, and the greater the resulting amplitude. This is why you push a child on a swing “in sync” with the swing’s motion. The resulting phenomenon is called resonance. When the driving and natural frequencies are the same, the result is called perfect resonance.

There are several famous/infamous cases of forced oscillations and resonance. In Equation 1, you see a movie of the Tacoma Narrows Bridge. A few months after it was built in 1940, strong winds caused the bridge to oscillate at its natural frequency, and the amplitude of the oscillations increased over time until the bridge collapsed. The precise cause of the collapse is a matter of some debate, but the resonant oscillations played a large part.

The Bay of Fundy in Nova Scotia provides another famous example. The tides vary greatly in the bay with the water level changing by as much as 16 meters. One reason for the dramatic tides is that the natural frequency of the bay, the time it takes for a wave to go from one end to the other, is close to the driving frequency of the tide cycle, which is about 12.5 hours.

As a third example, the natural frequency of one- to three-story buildings is close to the driving frequency supplied by some earthquakes, which is why these buildings (very common in San Francisco) often sustain the heaviest damage during quakes.

In Equation 2, you see a graph called a resonance curve. It is a graph of amplitude versus frequency for a system that has both a damping force and an external driving force. We call the natural (angular) frequency \( \omega_n \), and use \( \omega \) to indicate the driving frequency. As the driving frequency \( \omega \) approaches the natural frequency \( \omega_n \), the amplitude increases dramatically.

Natural frequencies can be “natural,” but in some cases they can also be controlled. Electric circuits, such as those used to tune radios to stations of different frequencies, are designed so that humans can change the natural frequency of the circuit. As you turn the radio dial, you are changing the natural frequency of the circuit. It then “tunes in” a driving frequency from a radio station that matches the natural frequency of the circuit. These concepts have entered everyday language. People say that “an idea resonates with me.” Such everyday speech is good physics; they mean the “driving frequency” of the idea is close to the “natural frequency” of their own beliefs.

15.29 - Gotchas

To calculate the amplitude of an object moving in SHM, measure the difference between two successive peaks of its graph. No, that is the period you just measured. The amplitude is the height of a peak of the graph above the horizontal (time) axis.

The slope at any point on the displacement graph of an object in SHM is its velocity. Yes, you are correct. This is a point that is true of any displacement graph, not just an SHM graph.
Simple harmonic motion (SHM) is a kind of repeated, consistent back and forth motion, like the swinging of a pendulum. It is caused by a restoring force that varies linearly with displacement.

The displacement associated with such motion can be described with a sinusoidal function, typically a cosine. The displacement is zero at equilibrium and maximum at the extreme positions.

Just as with other types of repetitive motion, the period of SHM is the amount of time required to complete one cycle of motion. The frequency is the number of cycles completed per second. It is the reciprocal of the period. The unit of frequency is the hertz (Hz), equal to one inverse second.

Angular frequency is the frequency measured in radians per second. It is represented by the Greek letter ω and is seen in the function for harmonic motion. If the object in simple harmonic motion is a mass on a spring, the spring constant and the mass determine the angular frequency.

The amplitude of harmonic motion is the maximum displacement from equilibrium. It is represented by A and appears as the coefficient of the cosine in the displacement function for SHM.

The phase constant, φ, specifies the position at the zero time, effectively shifting the graph to the left or right.

The velocity and acceleration functions for SHM are also sinusoidal. The maximum velocity occurs at equilibrium, and it is zero at the extremes. Acceleration is the opposite: zero at equilibrium and maximum at the extremes. These relationships follow from the general nature of velocity as the instantaneous slope of the displacement graph, and acceleration as the slope of velocity.

As an object like a mass on a spring moves in SHM, its total energy remains constant, although it transforms from potential to kinetic energy and back continuously. If you stretch the spring and release it to begin the motion, the amount of work you do on the spring is the amount of potential energy you have stored in it. All the energy is potential energy at the extremes, and kinetic energy at the equilibrium position.

A simple pendulum displays simple harmonic motion in its angular displacement, provided that the amplitude of the motion is small. Instead of a restoring force, there is a restoring torque due to gravity. The period of a pendulum depends upon the length of the pendulum and the acceleration of gravity.

The simple pendulum is a special case of the more complicated physical pendulum. In general, the period of a physical pendulum depends upon its moment of inertia, mass, and the distance from the pivot point to its center of mass, as well as the acceleration of gravity.

Sometimes a damping force opposes oscillatory motion. A typical damping force is proportional to the velocity of the object, which changes with time.

A force that acts with the restoring force can maintain or increase the amplitude of oscillations. Forced oscillations occur when such a driving force is present. The natural frequency of a system is the frequency at which it will oscillate in the absence of external force. As the frequency of the driving force approaches the natural frequency, energy is transferred more efficiently and the system’s oscillation amplitude increases. When these frequencies are approximately equal, resonance occurs.
Chapter 15 Problems

Chapter Assumptions

The general form of the equation of motion for an object in SHM is \( x(t) = A \cos(\omega t + \phi) \).

Conceptual Problems

C.1 A bouncing ball returns to the same height each time. Is this an example of simple harmonic motion? Explain your answer.
   - Yes   - No

C.2 Consider the displacement, velocity, and acceleration vectors of an object moving in SHM. Which pair of these vectors always point in opposite directions?
   - Displacement
   - Velocity
   - Acceleration

C.3 In old pocket watches, a balance wheel acts as a torsional pendulum, rotating with a fixed period. If a pocket watch is running slow, the period of the balance wheel is too long. Would you add or remove mass from the outer edge of the balance wheel to correct it?
   - Remove mass
   - Add mass

C.4 For a physical pendulum, what happens to the period as the pivot point gets very close to the center of mass? Justify your answer using the equation for the period of a physical pendulum.
   - i. The period gets longer
   - ii. The period shortens
   - iii. The period stays the same

C.5 What are the units for the damping coefficient constant?
   - N/m
   - kg/s
   - kg/s²

Section Problems

Section 0 - Introduction

0.1 Using the simulation in the interactive problem in this section, answer the following questions. (a) If you increase the amplitude, does the period increase, decrease, or stay the same? (b) What does the shape of the curve look like?
   (a) i. Increase
       ii. Decrease
       iii. Stay the same
   (b) i. Line
       ii. Parabola
       iii. Sinusoidal function
       iv. Circle

Section 3 - Period and frequency

3.1 Consider the minute hand on a clock. (a) Compute the frequency of its motion in cycles per second. State your answer to three significant digits. (b) Do the same for the hour hand.
   (a) _______ Hz
   (b) _______ Hz
3.2 A graph of the displacement of an object moving in SHM is shown. Determine the frequency of the object's motion. (Assume you can read the graph points to two significant figures.)

\[
\text{Hz}
\]

**Section 4 - Angular frequency**

4.1 What is the angular frequency of the second hand on a clock? (State your answer using three significant figures.)

\[
\text{rad/s}
\]

4.2 A potter's wheel rotates with an angular frequency of 1.54 rad/s. What is its period?

\[
\text{s}
\]

**Section 5 - Amplitude**

5.1 What is the amplitude of an object moving in SHM if its displacement in meters is described by:

(a) \( x(t) = 5 \cos(t - \pi/2) \)

(b) \( x(t) = 4 \cos(2\pi t) - \cos(2\pi t) \)

(c) \( x(t) = 4 \cos^2(\pi t) - 4 \sin^2(\pi t) \)

\[
\text{m}
\]

5.2 The displacement of an object moving in SHM is graphed as shown. What is its amplitude of motion? (Assume you can read the graph points to two significant figures.)

\[
\text{m}
\]

5.3 An object moving in SHM has an amplitude of 3.5 m and a period of 4.0 s. Which of the following equations could describe its displacement over time?

- \( x(t) = 2.0 \cos (3.5t) \)
- \( x(t) = 3.5 \cos (t + 2.0) \)
- \( x(t) = 3.5 \cos ((\pi/2)t + 2.0) \)
- \( x(t) = 3.5 \cos (2.0\pi t) \)

5.4 The displacement of an object in meters is described by the function \( x(t) = 4.4 \cos (7.4\pi t + \pi/2) \) where \( t \) is measured in seconds. What are the (a) amplitude, (b) frequency, (c) period of the object's motion?

\[
\text{m}
\]

\[
\text{Hz}
\]

\[
\text{s}
\]

**Section 6 - Interactive problem: match the curve**

6.1 Using the simulation in the interactive problem in this section, what (a) amplitude and (b) period should be used to match the graph?

\[
\text{m}
\]

\[
\text{s}
\]
Section 7 - Phase and phase constant

7.1 The displacement graph of an object moving in SHM is shown. Which of the following describe the phase \( \varphi \) for the object's motion?

- \( 0 < \varphi < \pi/2 \)
- \( \pi/2 < \varphi < \pi \)
- \( \pi < \varphi < 3\pi/2 \)
- \( 3\pi/2 < \varphi < 2\pi \)

7.2 An object moves in SHM with amplitude 4.37 m. The phase constant of the function describing its motion is 1.33 rad. What is the object's displacement at time \( t = 0 \) s?

\[ \text{m} \]

7.3 The graph of an object moving in SHM is shown. The amplitude of its motion is 2.7 m and at time \( t = 0 \) s, its displacement is \(-1.6\) m. What is the phase constant \( \varphi \)?

\[ \text{rad} \]

7.4 The graph of an object moving in SHM is shown. The amplitude of its motion is 3.6 m and at time \( t = 0 \) s, its displacement is 2.6 m. What is the phase constant \( \varphi \)? Give a positive answer between 0 and \( 2\pi \).

\[ \text{rad} \]

Section 8 - Sample problem: graph equation

8.1 An object moving in SHM has its most negative displacement at time \( t = 0 \) s. If the amplitude of the motion is 7.8 m and the angular frequency is 5.6 rad/s, which of these equations describes the motion?

- \( x(t) = 7.8 \cos (5.6t) \)
- \( x(t) = 7.8 \cos (5.6\pi t) \)
- \( x(t) = 7.8 \cos (5.6t + \pi/2) \)
- \( x(t) = 7.8 \cos (5.6t + \pi) \)

8.2 An object moving in SHM has zero displacement at time \( t = 0 \) s, and positive displacement a moment after. If the amplitude of the motion is 4.8 m and the frequency is 4.6 Hz, which of these equations describes the motion?

- \( x(t) = 4.8 \cos (9.2t + 3\pi/2) \)
- \( x(t) = 4.8 \cos (9.2\pi t + \pi/2) \)
- \( x(t) = 4.8 \cos (9.2t + 3\pi/2) \)
- \( x(t) = 4.8 \cos (4.8t + \pi/2) \)

8.3 An object moving in SHM has maximum (positive) displacement at time \( t = 0 \) s. If the amplitude of the motion is 2.1 m and the period is 6.4 s, what is the displacement at time \( t = 2.8 \) s?

\[ \text{m} \]

8.4 An object moving in SHM has zero displacement at time \( t = 0 \) s, and negative displacement a moment after. If the amplitude of the motion is 6.8 m and the frequency is 4.6 Hz, what is the displacement at time \( t = 3.4 \) s?

\[ \text{m} \]

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Section 9 - Velocity

9.1 In a car engine, a piston moves in SHM with an amplitude of 0.410 m. The engine is running at 2400 rpm, which is an angular frequency of 251 rad/s. What is the maximum speed of the piston?

_________ m/s

9.2 The equation for the displacement in meters of an object moving in SHM is \( x(t) = 1.50 \cos (4.20t) \) where \( t \) is in seconds. (a) What is the maximum speed of the object? (b) At what time does it first reach the maximum speed?

(a) _________ m/s
(b) _________ s

Section 11 - Acceleration

11.1 A ball on a spring moves in SHM. At time \( t = 0 \) s, its displacement is 0.50 m and its acceleration is \(-0.72 \text{ m/s}^2\). The phase constant for its motion is 0.84 rad. What is the ball's displacement at \( t = 3.4 \) s?

_________ m

11.2 A platform moves up and down in SHM, with amplitude 0.050 m. Resting on top of the platform is a block of wood. What is the shortest period of motion for the platform so that the block will remain in constant contact with it?

_________ s

Section 13 - Summary of simple harmonic motion

13.1 A particle moves in SHM, with displacement defined by the equation \( x(t) = 0.0018 \cos (2\alpha t) \), where \( x \) is measured in meters and \( t \) in seconds. (a) What is the particle's maximum speed? (b) What is its maximum acceleration? (c) What is its acceleration when \( x = 0.0013 \) m?

(a) _________ m/s
(b) _________ m/s²
(c) _________ m/s²

13.2 A block attached to a spring moves in SHM on a frictionless surface. The acceleration of the block is given by the equation \( a(t) = -3.6x(t) \), where \( x \) is measured in meters and \( t \) in seconds. (a) What is the angular frequency of the block's motion? (b) When the block has maximum acceleration, its displacement is \(-2.3 \) m. What is the amplitude of the block's motion?

(a) _________ rad/s
(b) _________ m

13.3 A particle is moving in SHM. Its maximum acceleration is \(2.27 \text{ m/s}^2\) and its maximum velocity is \(1.39 \text{ m/s}\). What is the period of its motion?

_________ s

13.4 The acceleration of an object moving in SHM is defined by the equation \( a(t) = -3.72x(t) \), where \( x \) is measured in meters and \( t \) in seconds. At time \( t = 0 \) s, the object has its maximum positive displacement. (a) What is the first time after \( t = 0 \) s that the object is at the equilibrium position (zero displacement)? (b) In what direction is the object moving at that time?

(a) _________ s
(b)  □ Positive direction
  □ Negative direction

Section 14 - Simple harmonic motion and uniform circular motion

14.1 A particle moves in a circle of radius 0.45 m at a constant speed, completing one revolution every 1.2 s. The particle is on the positive x axis at time \( t = 0 \) s. Write an equation for the y displacement of the particle as a function of time.

  □ \( y(t) = 0.45 \cos ((2\pi/1.2)t) \)
  □ \( y(t) = 0.45 \sin ((2\pi/1.2)t) \)
  □ \( y(t) = 0.45 \cos (1.2t) \)
  □ \( y(t) = 0.45 \sin (1.2t) \)

14.2 The text states an equation for the x displacement of an object in uniform circular motion: \( x(t) = A \cos \omega t \). Write a corresponding function for the y displacement, also using the cosine function.

  □ \( y(t) = 2A \cos \omega t \)
  □ \( y(t) = A \cos (\omega t - \pi) \)
  □ \( y(t) = A \cos (\omega t - \pi/2) \)
  □ \( y(t) = A \cos (\omega t - 3\pi/4) \)

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Section 15 - Period, spring constant, and mass

15.1 A block with mass 0.55 kg on a frictionless surface is attached to a spring with spring constant 41 N/m. The block is pulled from the equilibrium position and released. What is the period of the system?

\[ \text{\_\_\_\_\_\_ s} \]

15.2 A 0.683 kg mass moves in SHM at the end of a spring. It takes 1.41 s to move from the position with the spring fully extended to the position with the spring fully compressed. What is the spring constant?

\[ \text{\_\_\_\_\_\_ N/m} \]

15.3 A block with mass 0.382 kg is attached to a horizontal spring with spring constant \( k = 1.28 \) N/m on a frictionless surface. The block is pulled 0.753 m from equilibrium and released. (a) What is the amplitude of the block’s motion? (b) What is its period? (c) How long after release does the block take to first return to its equilibrium position? (d) What is its speed at that position?

(a) \[ \text{\_\_\_\_\_\_ m} \]
(b) \[ \text{\_\_\_\_\_\_ s} \]
(c) \[ \text{\_\_\_\_\_\_ s} \]
(d) \[ \text{\_\_\_\_\_\_ m/s} \]

15.4 A 1.20 kg mass is attached to the end of a spring of unknown spring constant. The spring is compressed a distance of 0.300 meters, and when it is released, the mass oscillates horizontally on a frictionless surface. When the mass is 0.100 m from equilibrium, it is seen to be moving at a speed of 2.00 m/s. (a) What is the amplitude of the motion? (b) Find the spring constant, \( k \). (c) What is the maximum speed of the mass?

(a) \[ \text{\_\_\_\_\_\_ m} \]
(b) \[ \text{\_\_\_\_\_\_ N/m} \]
(c) \[ \text{\_\_\_\_\_\_ m/s} \]

15.5 A car has a mass of 1550 kg. Its four passengers have a combined mass of 322 kg. The car's suspension has four identical springs. The suspension frequency for a comfortable ride is 1.30000111474276 Hz. Assume the load is distributed evenly. What should the spring constant for each spring be?

\[ \text{\_\_\_\_\_\_ N/m} \]

15.6 A spring of unknown spring constant is attached to a 2.30 kg mass. The mass is pulled horizontally outward by a distance “\( A \)” from equilibrium, then released at \( t = 0 \). At \( t = 0.200 \) seconds, the mass first reaches the equilibrium position, where it is seen to be moving at a velocity of \( v = -0.950 \) m/s. (a) What is the period of the motion? (b) What is the frequency of the motion? (c) Calculate the spring constant of the spring. (d) Calculate the amplitude of motion, \( A \).

(a) \[ \text{\_\_\_\_\_\_ s} \]
(b) \[ \text{\_\_\_\_\_\_ Hz} \]
(c) \[ \text{\_\_\_\_\_\_ N/m} \]
(d) \[ \text{\_\_\_\_\_\_ m} \]

15.7 A large cube made of cork floats in a swimming pool, with one face parallel to the water surface. The cube is uniform in density, with mass 11.6 kg and each edge 0.381 m in length. The top surface of the cube is briefly tapped, causing it to oscillate in the vertical direction in SHM. The cube is damp, of course, but its motion is not damped. What is the period of the cube's motion? Use a value of 997 kg/m$^3$ for the density of the water, an appropriate value for the water in a swimming pool at 25°C.

\[ \text{\_\_\_\_\_\_ s} \]

Section 16 - Interactive problem: match the curve again

16.1 A block at the end of a horizontal spring moves in SHM as shown by the graph. The mass of the block is 0.38 kg. Assume you can read the graph points to two significant figures. (a) What is the spring constant? (b) What is the block’s maximum acceleration?

(a) \[ \text{\_\_\_\_\_\_ N/m} \]
(b) \[ \text{\_\_\_\_\_\_ m/s^2} \]

16.2 A block on a spring moves in simple harmonic motion. The mass of the block is 0.30 kg and the spring constant is 0.80 N/m. If the block is at its maximum positive displacement of 0.70 m at time \( t = 0 \) s, draw a graph of the block’s displacement over the first three seconds.
16.3 Using the simulation in the interactive problem in this section, what should the (a) amplitude and (b) the mass of the puck be in order to match the graph?

(a) \( \text{m} \)

(b) \( \text{kg} \)

Section 18 - Work and the potential energy of a spring

18.1 A motor pulls a spring 0.73 m away from its equilibrium position, doing 3.5 J of work on the spring in the process. What is the spring constant?

\( \text{N/m} \)

18.2 A spring with a spring constant of 45 N/m is pulled 1.4 m away from its equilibrium position. How much potential energy is stored in the spring?

\( \text{J} \)

Section 19 - Total energy

19.1 A block with mass 0.67 kg, resting on a horizontal frictionless surface, is attached to the end of a spring with spring constant 49 N/m. The block is released from rest at a distance 0.035 m from the equilibrium position. What is the total energy of the system?

\( \text{J} \)

19.2 A block of mass 0.35 kg oscillates in SHM on a spring with an amplitude of 0.96 m. The maximum acceleration of the block is 2.7 m/s\(^2\). What is the total energy of the system?

\( \text{J} \)

19.3 An object is moving in simple harmonic motion with amplitude 0.28 m and total energy 2.5 J. By some means its amplitude is increased to 0.34 m. What is the total energy of the system now?

\( \text{J} \)

19.4 A block and spring system oscillating in SHM has a maximum speed of 1.18 m/s and a total energy of 1.71 J. What is the mass of the block?

\( \text{kg} \)

Section 20 - Interactive checkpoint: spring energy and period

20.1 A 4.6 kg block is attached to a spring with a spring constant of \( k = 6.3 \text{ N/m} \). It oscillates in SHM with an amplitude of 1.8 m. What is the maximum speed of the block?

\( \text{m/s} \)

20.2 A block on a spring moves in SHM with amplitude 0.23 m and period 3.8 s. The mass of the block is 2.1 kg. What is the total energy of the system?

\( \text{J} \)

Section 21 - Sample problem: falling block on a spring

21.1 A 2.7 kg block hangs from a vertical spring whose upper end is fixed. The spring constant is 93 N/m. Define the \( PE \) of the system to be zero when the block is at the equilibrium position. The block is then set into motion and oscillates with an amplitude of 3.8 m. When the block is at its lowest position, what is the potential energy stored in the spring?

\( \text{J} \)

21.2 A 3.5 kg block oscillates in SHM from the end of a vertical spring with spring constant 88 N/m. The block's maximum speed is 12 m/s. What is the block's amplitude of motion?

\( \text{m} \)

21.3 An object is hung at the end of a vertical unstretched spring, and released. It falls 0.347 m before reversing direction. What is its period of motion?

\( \text{s} \)

21.4 A 2.7 kg block hangs at the end of a spring. When the block is at rest, the spring is stretched 0.38 m away from its equilibrium position. The block is set into simple harmonic motion and oscillates with an amplitude of 0.24 m. What is the total energy of the block/spring system?

\( \text{J} \)
Section 22 - A torsional pendulum

22.1 An irregularly-shaped 1.4 kg object is suspended from a wire with a known torsion constant of 0.49 N·m/rad. The object's period is 1.2 seconds. What is the object's moment of inertia for rotations about this axis?

_______ kg · m²

22.2 A thin square slab of material is suspended "on edge" at the end of a torsion pendulum, so that the axis of rotation passes through the center of the square, parallel to an edge. The mass of the slab is 0.78 kg and the length of an edge is 0.28 m. The torsion constant of the wire is 6.2 N·m/rad. What is the period of motion when the slab oscillates?

_______ s

Section 23 - A simple pendulum

23.1 You need to know the height of a room, but you have no tape measure. You fasten one end of a string to the ceiling of the room, and tie a small rock at the other end so it almost touches the floor. You start this simple pendulum swinging slightly, and measure its period, which is 3.56 seconds. How tall is the room?

_______ m

23.2 On the moon of a distant planet, an astronaut measures the period of a simple pendulum, 0.85 m long, and finds it is 4.7 seconds. Back on Earth, she could throw a rock 13 m straight up (while wearing her spacesuit). With the same effort, how far up can she throw the same rock at her present location? Ignore the effects of air resistance.

_______ m

23.3 It is the year 2305 and the tallest structure in the world has an insane height of 3.19×10⁶ m above the surface of the Earth. A pendulum clock that keeps perfect time on the surface of the Earth is placed at the top of the tower. How long does the clock take to register one elapsed hour? The radius of the Earth is 6.38×10⁶ m and its mass is 5.97×10²⁴ kg.

_______ minutes

Section 24 - Interactive problem: a pendulum

24.1 Using the simulation in the interactive problem in this section, what is the length of string needed to achieve the desired period for the pendulum?

_______ m

Section 25 - Period of a physical pendulum

25.1 A flat circular disk with diameter 0.36 m and mass 0.43 kg is suspended so it can swing freely from a pivot 0.12 m from its center. (The axis of rotation is perpendicular to the plane of the disk.) What is its period of oscillation?

_______ s

25.2 A small hole is drilled in a meter stick is to act as a pivot. The meter stick swings in a short arc as a physical pendulum. How far from the center of mass should the pivot point be for a period of 2.00 seconds?

_______ m

Section 26 - Sample problem: meter-stick pendulum

26.1 A thin square slab of material is suspended so that it can rotate freely around one of its edges, which is parallel to the ground. The slab is uniform in density and has mass 0.78 kg. The length of each side is 0.67 m. What is the slab's period?

_______ s
26.2 A disk with diameter 4.30 m is allowed to oscillate freely as a physical pendulum about an axis that is perpendicular to its surface, and located some distance from its center. If its period of oscillation is 3.50 s, what is the minimum possible distance of the axis from the center of the disk?

_______ m

26.3 A solid spherical ball of mass 0.227 kg is suspended from a thin rod of negligible mass to form a pendulum. The diameter of the ball is 0.214 m and the length of the rod is 0.125 m. (a) If you considered this to be a simple pendulum, with length equal to the rod length plus the radius of the ball, what would its period be? (b) What is the actual period of the pendulum?

(a) _______ s
(b) _______ s